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The role of the η' in determinations of the light quark masses

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Abstract: The corrections to the current algebra mass formulae for the pseudoscalar mesons are analyzed by means of a simultaneous expansion in powers of the light quark masses and powers of $1/N_c$. The relative magnitude of the two expansion parameters is related to the mass ratio $M_\eta^2/M_{\eta'}^2$ and to the mixing angle $\theta_{\eta\eta'}$, which both represent quantities of order $N_c m_q$. A set of mass formulae is derived, including an inequality, which leads to bounds for the ratios m_u/m_d and m_s/m_d. Also, it is shown that the decay $\eta \to 3\pi$ represents a very sensitive probe of chiral symmetry breaking. If the experimental situation can be improved, this process will yield a clean direct measurement of the ratio $(m_d^2 - m_u^2)/m_s^2$.

Keywords: QCD, quark masses, large N_c expansion, effective Lagrangian

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I. Introduction

Since many years, Haridas Banerjee is fascinated by the peculiar role of the η' in connection with the U(1)-anomaly. In particular, the topological implications for the gluon field configurations which characterize the QCD vacuum form one of the foci of his attention [1]. I am confident, therefore, that the following discussion of the role of the η' in the low energy structure of the theory will arouse his interest.

The low energy properties of QCD are governed by an approximate, spontaneously broken symmetry, which originates in the fact that three of the quarks happen to be light. If m_u, m_d, m_s are turned off, the symmetry becomes exact. The spectrum of the theory then contains eight strictly massless pseudoscalar mesons, the Goldstone bosons connected with the spontaneous symmetry breakdown.

In the limit where the number of colours is taken large, the quark loop graph which gives rise to the anomaly in the divergence of the singlet axial current is suppressed [2]. At leading order in an expansion in powers of $1/N_c$, the theory thus aquires an additional symmetry, which is also spontaneously broken [3]. The spectrum of QCD, therefore, contains a ninth state, the η', which becomes massless if not only m_u, m_d, m_s are turned off, but if in addition the number of colours is sent to infinity. The purpose of this paper is to study the mass spectrum of the pseudoscalars within a framework which accounts for contributions of leading and first nonleading order in the expansion in powers of $1/N_c$.

I use the effective Lagrangian method, which describes the low energy structure of the theory in terms of an expansion in powers of energies, momenta and quark masses [4, 5]. The application of this method in the large N_c limit is discussed in detail in the literature [6]. I first review the derivation of the leading terms in some detail and only then take up the discussion of the higher order terms in the derivative expansion and their implications for the pattern of the pseudoscalar masses, which to my knowledge is new.

The relevant dynamical variable is the Goldstone field $U(x)$, which lives on the quotient G/H, where G and H are the (approximate) symmetry groups of the Hamiltonian and of the ground state, respectively. Since I wish to study the large N_c limit, G is the group $\mathrm{U}(3)_R \times \mathrm{U}(3)_L$ and $U(x) \in \mathrm{U}(3)$. The unimodular part of the field $U(x)$ contains the degrees of freedom of the pseudoscalar octet, while the phase $\det U(x) = e^{i\phi_0(x)}$ describes the η'.

The effective Lagrangian is formed with the field $U(x)$ and its derivatives, $\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, \ldots, m)$. It contains the quark mass matrix

$$m = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

and depends on the number of colours. Disregarding the vacuum energy of the gluon field which generates a cosmological constant of order N_c^2, the expansion in inverse powers of N_c starts with a term of order N_c,

$$\mathcal{L}_{eff} = \mathcal{L}_{[1]} + \mathcal{L}_{[0]} + \mathcal{L}_{[-1]} + \ldots$$

II. Structure of the effective Lagrangian in the limit $N_c \to \infty$

The leading term $\mathcal{L}_{[1]}$ does not know about the occurrence of an anomaly. It is invariant under the full group of chiral rotations, except for the explicit symmetry breaking generated by the quark masses. Under the action of $\mathrm{U}(3)_R \times \mathrm{U}(3)_L$, the effective field transforms according to $U(x) \to V_R\, U(x) V_L^\dagger$, with $V_R \in \mathrm{U}(3)_R$, $V_L \in \mathrm{U}(3)_L$. The only function of $U(x)$ which is invariant under this operation is a constant. Invariants can only be formed with the derivatives of the field. At second order in the derivative expansion, there are two independent invariants. Abbreviating traces with the symbol $\langle \ldots \rangle$, they are given by $\langle \partial_\mu U^\dagger \partial^\mu U \rangle$ and $\langle U^\dagger \partial_\mu U \rangle \langle U^\dagger \partial^\mu U \rangle$. The second one involves two traces. It can only arise from graphs with two or more quark loops and is thus suppressed by one power of $1/N_c$. Accordingly, it does not belong to $\mathcal{L}_{[1]}$, but occurs among the corrections of first nonleading order, described by $\mathcal{L}_{[0]}$. There is an infinite string of such invariants, which may be ordered according to the number of derivatives. In particular, there are three independent invariants with four derivatives, viz. $\langle \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U \rangle$, $\langle \partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U \rangle$ and $\langle \partial_\mu U^\dagger \partial^\mu U \rangle^2$. Again, the third one does not appear in $\mathcal{L}_{[1]}$, because it involves two traces.

The explicit symmetry breaking due to the quark masses is readily accounted for. Under chiral rotations, the mass matrix m transforms in the same manner as the meson field, viz. $m \to V_R\, m V_L^\dagger$. The symmetry breaking term of lowest dimension is given by the invariant $\langle U^\dagger m \rangle +$ c.c. Since this expression involves a single trace, it is not suppressed in the large N_c limit and thus shows up in $\mathcal{L}_{[1]}$. There are again three invariants involving the second power of the mass matrix. Two of these are suppressed, however, because they involve two traces. Only the third term, $\langle U^\dagger m U^\dagger m \rangle +$ c.c., belongs to $\mathcal{L}_{[1]}$. Also, one may write down a term involving two derivatives and one mass matrix, viz. $\langle \partial_\mu U^\dagger \partial^\mu U\, U^\dagger m \rangle +$ c.c. and so on. The effective Lagrangian $\mathcal{L}_{[1]}$ collects all of the invariants which can be formed with $m, U(x), \partial U(x), \ldots$ and which involve a single trace.

This property of QCD may readily be formulated at the level of the effective theory. The presence of the additional coupling constant also shows up in the effective Lagrangian, $\mathcal{L}_{eff} \to \overline{\mathcal{L}}_{eff} = \overline{\mathcal{L}}_{eff}(U, m, \theta)$. The expression must have the property that the transformation

$$U' = V_R U V_L^\dagger \ , \quad m' = V_R m V_L^\dagger \ , \quad \theta' = \theta - \alpha$$

leaves it invariant, $\overline{\mathcal{L}}_{eff}(U', m', \theta') = \overline{\mathcal{L}}_{eff}(U, m, \theta)$.[1]

The consequences for the structure of the effective Lagrangian are the following. Since the phase of the determinant $e^{i\phi_0} = \det U$ transforms according to $\phi_0' = \phi_0 + \alpha$, the combination $\phi_0 + \theta$ as well as the derivatives thereof remain invariant. This implies that the effective Lagrangian may again be represented as a superposition of the invariants which can be formed with $m, U, \partial U, \ldots$, but the coefficients now depend on $\phi_0 + \theta, \partial_\mu \phi_0, \ldots$ Up to and including terms with two derivatives or one factor of m, the most general expression consistent with the symmetries of the QCD Hamiltionian is

$$\mathcal{L}_{eff} = c_0 + c_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle + c_2 \langle U^\dagger m \rangle + c_2^\star \langle m^\dagger U \rangle + c_3 \partial_\mu \phi_0 \partial^\mu \phi_0 \ldots \ ,$$

with $c_n = c_n(\theta + \phi_0)$. The expression is valid to all orders in $1/N_c$. The symmetry does not constrain the form of the coefficient functions, except that c_0, c_1, c_3 must be real and even under parity, $c_n(-x) = c_n(x)$, while c_2 obeys $c_2(-x) = c_2(x)^\star$. In particular, the argument given in the preceding section, according to which terms without derivatives or quark mass factors are independent of the field $U(x)$ only holds at leading order in $1/N_c$. The general statement, valid to all orders, is that these terms are of the form $c_n(\phi_0 + \theta)$ - they involve the field $U(x)$ exclusively through the phase ϕ_0 of $\det U$.

IV. Dependence on θ

[1]More precisely, the corresponding action is invariant - the effective Lagrangian changes by a total derivative. The phenomenon gives rise to the Wess-Zumino-term. Since the corresponding vertices involve five or more meson fields, they do not contribute to either the mass formulae or to the decay $\eta \to 3\pi$ and I therefore disregard the complication.

The reason why the various coefficients reduce to constants in the limit $N_c \to \infty$ is that, in this limit, the dependence of the various Green functions and matrix elements on θ is suppressed [2]. Compared to the leading contribution of the QCD Lagrangian, $-G^a_{\mu\nu}G^{a\mu\nu}/4g^2$, the term $-\theta\, G^a_{\mu\nu}\tilde{G}^{a\mu\nu}/32\pi^2$ represents a small perturbation of order $g^2\theta \sim \theta/N_c$. The gluonic contribution to the vacuum energy density, e.g., is of the form $\epsilon_g = N_c^2 g_0(\theta/N_c) + N_c g_1(\theta/N_c) + \ldots$ The θ-dependence of the coefficients occuring in the effective Lagrangian is of the same structure,

$$c_n = (N_c)^{p_n}\{c_n^0(\tilde{\theta}) + N_c^{-1}c_n^1(\tilde{\theta}) + \ldots\} \;, \quad \tilde{\theta} = \frac{\phi_0 + \theta}{N_c} \;,$$

with $p_0 = 2$, $p_1 = p_2 = 1$, $p_3 = 0, \ldots$ In fact, the leading term in c_0 differs from the gluonic vacuum energy only in sign, $c_0^0(x) = -g_0(x)$.

Since symmetry implies that the variable ϕ_0 only enters in the combination $\phi_0 + \theta$, the suppression of the θ-dependence implies that ϕ_0 only occurs at nonleading orders of the $1/N_c$ expansion, in agreement with the form of the leading order effective Lagrangian given in section II. At first nonleading order, the Lagrangian picks up a contribution from the expansion of c_0, proportional to $(\phi_0+\theta)^2$, and a term of order $\phi_0+\theta$ from the coefficient c_1,

$$\mathcal{L}_{[0]} = K_0(\phi_0 + \theta)^2 + iK_1(\phi_0 + \theta)\langle U^\dagger\chi - \chi^\dagger U\rangle + K_2\partial_\mu\phi_0\partial^\mu\phi_0 + \ldots$$

The effective coupling constants K_0, K_1, K_2 occurring here are quantities of order N_c^0. At low energies, the first term is the most important one. It represents the contribution of order θ^2 in the expansion of the gluonic vacuum energy, which is referred to as the topological susceptibility,

$$= (-i)\int dx\langle 0|T\omega(x)\omega(y)|0\rangle_g \;.$$

In this notation, $K_0 = -\frac{1}{2}\tau$. Among the corrections containing two derivatives or one quark mass factor, there is a term proportional to $\langle\partial_\mu U^\dagger\partial^\mu U\rangle$, which renormalizes the pion decay constant through a correction of order $1/N_c$, as well as one $\sim \langle U^\dagger\chi + \chi^\dagger U\rangle$, which renormalizes the constant B. Including these in the leading terms and switching the perturbation θ off, the final result for the effective Lagrangian then reads

$$\begin{aligned}\mathcal{L}_{eff} = &\tfrac{1}{4}F^2\langle\partial_\mu U^\dagger\partial^\mu U + U^\dagger\chi + \chi^\dagger U\rangle - \tfrac{1}{2}\tau\phi_0^2 \\ &+L_5\langle\partial U^\dagger\partial U(U^\dagger\chi + \chi^\dagger U)\rangle + L_8\langle U^\dagger\chi U^\dagger\chi + \chi^\dagger U\chi^\dagger U\rangle \\ &+iK_1\phi_0\langle U^\dagger\chi - \chi^\dagger U\rangle + K_2\partial_\mu\phi_0\partial^\mu\phi_0 \ .\end{aligned} \tag{1}$$

The terms listed are the leading and first nonleading contributions in a simultaneous expansion in powers of $1/N_c$, in the derivatives of the field $U(x)$ and in powers of the quark mass matrix. The terms proportional to F^2 are of order $N_c\partial^2$ and $N_c m$, while the susceptibility term is of order $N_c^{\,0}$. The relative size of these contributions depends on the relative magnitude of the three expansion parameters. The remainder represents a correction, suppressed either by two powers of momentum, by a factor of m or by a factor of $1/N_c$ as compared to one of the three leading terms: the contributions proportional to L_5, L_8, K_1 and K_2 are of order $N_c\partial^2 m, N_c m^2, N_c^{\,0} m$ and $N_c^{\,0}\partial^2$, respectively.

V. Mass spectrum of the pseudoscalars at leading order

If the electromagnetic interaction is turned off, the masses of the pseudoscalars only depend on the scale Λ_{QCD} and on the quark masses. At leading order in the combined expansion in powers of m and $1/N_c$, the masses are determined by the expression in the first line of equation (1), i.e. by the constant B contained in $\chi = 2Bm$ and by the topological susceptibility τ. Setting $U = \exp i\varphi/F$, we have $\phi_0 = \langle\varphi\rangle/F$, so that the terms quadratic in φ are $\frac{1}{4}\langle\partial_\mu\varphi\partial^\mu\varphi\rangle - \frac{1}{4}\langle\chi\varphi^2\rangle - \frac{1}{2}\tau\langle\varphi\rangle^2/F^2$. For those fields which carry electric charge or strangeness, this expression is diagonal and leads to the standard current algebra mass formulae, $M^2_{\pi^+} = (m_u + m_d)B$, $M^2_{K^+} = (m_u + m_s)B$, $M^2_{K^0} = (m_d + m_s)B$.

The states π^0, η and η' undergo mixing. The mixing angles between the π^0 and η, η' are proportional to the isospin breaking mass difference $m_d - m_u$. These angles are small, but play a crucial role e.g. for the transition $\eta \to 3\pi$. In the masses of the neutral particles, however, isospin breaking only shows up through contributions of order $(m_d - m_u)^2$, which are negligibly small. Disregarding these, the π^0 is degenerate with $\pi^\pm$, while the masses M_η and $M_{\eta'}$ are obtained by setting $\varphi = \varphi_8\lambda_8 + \varphi_9\sqrt{\frac{2}{3}}$ and diagonalizing the above quadratic form. Using the current algebra relations to express the quark masses in terms of M_K^2, M_π^2, the quadratic part of the Lagrangian takes the form

$$\mathcal{L}_{eff} = \tfrac{1}{2}(\partial\phi_8^2 + \partial\phi_9^2) - \tfrac{1}{2}(m_0^2\varphi_8^2 - 2\sigma_0\varphi_8\varphi_9 + M_0^2\varphi_9^2)$$
$$m_0^2 = \tfrac{1}{3}(4M_K^2 - M_\pi^2)\,,\ \sigma_0 = \tfrac{2}{3}\sqrt{2}(M_K^2 - M_\pi^2)\,,\ M_0^2 = 6\frac{\tau}{F^2} + \tfrac{1}{3}(2M_K^2 + M_\pi^2)\,.$$

Unfortunately, the topological susceptibility cannot yet be determined to sufficient precision, so that M_0 must be treated as an unknown. Since the sum of the two eigenvalues must agree with the trace of the matrix, we have $M_\eta^2 + M_{\eta'}^2 = m_0^2 + M_0^2$. Hence the topological susceptibility must approximately be equal to $\tau = \frac{1}{6}F^2(M_{\eta'}^2 + M_\eta^2 - 2M_K^2) = (180\,\mathrm{MeV})^4$ [6]. Setting

$$\varphi_8 = \cos\theta_{\eta'\eta}\,\eta + \sin\theta_{\eta'\eta}\,\eta'\ ,\quad \varphi_9 = -\sin\theta_{\eta'\eta}\,\eta + \cos\theta_{\eta'\eta}\,\eta'\ ,$$

the above quadratic form is diagonalized with

$$\sin 2\,\theta_{\eta'\eta} = -\tfrac{4}{3}\sqrt{2}\,\frac{M_K^2 - M_\pi^2}{M_{\eta'}^2 - M_\eta^2}\ . \tag{2}$$

The relation yields $\theta_{\eta'\eta} \simeq -22°$, in reasonable agreement with what is found phenomenologically [8]. The scheme, however, also implies that the two eigenvalues are related by $(m_0^2 - M_\eta^2)(M_{\eta'}^2 - m_0^2) = \sigma_0^2$. Solving for M_η^2 this gives

$$M_\eta^2 = \tfrac{1}{3}(4M_K^2 - M_\pi^2) - \tfrac{8}{3}\frac{(M_K^2 - M_\pi^2)^2}{(3M_{\eta'}^2 + M_\pi^2 - 4M_K^2)}\ . \tag{3}$$

If the second term is omitted, the relation reduces to the Gell-Mann-Okubo formula, which predicts $M_\eta \simeq 566\,\mathrm{MeV}$, slightly larger than what is observed. The second term indeed lowers the result, but the shift is much too large: While the Gell-Mann-Okubo prediction for M_η^2 only differs from the experimental value by 7 %, the repulsion generated by mixing now yields a value which is too low by about 20 %. The relation requires the η to be essentially degenerate with the K.

The problem originates in the fact that one is dealing with small effects here. The same framework also predicts that F_K is equal to F_π, while, experimentally, the two quantities differ by the factor 1.22. There is no reason why in the case of the masses, the corrections generated by nonleading orders of the expansion should be smaller.

The conclusions to draw from the above lowest order formulae are:

(a) The ratio $M^2_{\eta'}/M^2_\eta$ is about equal to 3, indicating that the breaking of $\mathrm{U}(3)_R\times\mathrm{U}(3)_L$-symmetry generated by the anomaly is larger than the breaking due to m_s by roughly this factor. The topological susceptibility which describes the effects of the anomaly in the framework of the effective Lagrangian and is of order N_c^0 is more important than the terms of order mN_c, which account for the symmetry breaking generated by the quark masses.

(b) The transition matrix elements of $\bar{q}mq$ between octet and singlet are of the same order of magnitude as the diagonal matrix elements. As a consequence, the mixing is suppressed only by about a factor of 3, such that the levels must repel quite strongly. The repulsion is described by the second term of eq.(3).

(c) This equation only accounts for those deviations from the Gell-Mann-Okubo formula, which arise from the interference with the η', while it neglects all other effects of order m^2. The observed pattern of masses requires that these other effects generate comparable contributions of opposite sign.

Indeed, the particle data table shows that the η' is nearly degenerate with the scalar nonet. These states do not undergo mixing with pseudoscalar one-particle states, but with the ground state as well as with two-particle states [9]. The corresponding effect on the mass of the pseudoscalars is of order m^2. It is suppressed by a relatively large energy denominator, because the masses of the scalars are large compared to those of the pseudoscalar octet, but the energy denominator is essentially the same as the one which suppresses the shift generated by mixing with the η'.

In the effective Lagrangian set up in the preceding section, the relevant terms are those proportional to L_5 and L_8. Their effect on the mass spectrum of the pseudoscalars is readily worked out, together with the corrections generated by K_1 and K_2 and I now turn to that calculation.

VI. Second order mass formulae

For the charged states, the Lagrangian remains diagonal. The current algebra mass formula for M_{π^+} is replaced by

$$M^2_{\pi^+} = (m_u + m_d)B\{1 + (m_u + m_d)\frac{8B}{F^2}(2L_8 - L_5)\}$$

and analogously for $M^2_{K^+}, M^2_{K^0}$. These relations agree with those of chiral perturbation theory [5], with two simplifications: (i) The coupling constants L_4 and L_6 do not occur here, because they are suppressed by one power of N_c. (ii) For the same reason, the chiral logarithms generated by the one loop graphs are absent. Since these graphs are inversely proportional to F^2, they only show up at the next order of the expansion under consideration.

As is well known, the above mass formulae imply that the corrections in the two mass ratios[2]

$$\frac{M_K^2}{M_\pi^2} \quad \frac{m_s + \hat{m}}{2\hat{m}}\{1 + \Delta_M\} \tag{4}$$

$$\frac{M^2_{K^0} - M^2_{K^+}}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\{1 + \Delta_M\} \tag{5}$$

are the same. Eliminating the quark masses in favour of M_π, M_K, the explicit expression for the correction becomes

$$\Delta_M = \frac{8}{F^2}(M_K^2 - M_\pi^2)(2L_8 - L_5) \ . \tag{6}$$

In the double ratio

$$Q^2 \equiv \frac{M_K^2}{M_\pi^2}\,\frac{M_K^2 - M_\pi^2}{M^2_{K^0} - M^2_{K^+}}$$

[2]The quantity $\hat{m}$ denotes the mean mass of u and d, $\hat{m} \equiv \frac{1}{2}(m_u + m_d)$

the correction drops out. Accordingly, the corresponding double ratio of quark masses is determined by the masses of the pseudoscalars

$$\frac{m_s^2 - \frac{1}{4}(m_u + m_d)^2}{m_d^2 - m_u^2} = Q^2 \tag{7}$$

up to and including first order corrections [5, 10]. Note that I have disregarded the electromagnetic interaction. Evaluating the corresponding self energies with the Dashen theorem [11], one finds $Q \simeq 24$. I will discuss the value of Q in more detail below, in connection with η-decay.

In the neutral sector, the contributions from L_5 and K_2 introduce off-diagonal elements into the the kinetic term. These are removed with the change of variables $\varphi \to \varphi - 4\{m, \varphi\}L_5 B/F^2 - 2\langle\varphi\rangle K_2/F^2$. The quadratic part of the Lagrangian then takes the same form as before, but the coefficients of the quadratic form are modified. The quantities m_0^2, σ_0 and M_0^2 are replaced by

$$\begin{aligned} m_1^2 &= \tfrac{1}{3}(4M_K^2 - M_\pi^2) + \tfrac{4}{3}(M_K^2 - M_\pi^2)\Delta_M \\ \sigma_1 &= \tfrac{2}{3}\sqrt{2}(M_K^2 - M_\pi^2)\{1 + \Delta_M - \Delta_N\} \\ M_1^2 &= 6\frac{\overline{\tau}}{F^2} + \tfrac{1}{3}(2M_K^2 + M_\pi^2)(1 - 2\Delta_N) + \tfrac{2}{3}(M_K^2 - M_\pi^2)\Delta_M \end{aligned} \tag{8}$$

Remarkably, the expression for m_1^2 only involves the same correction Δ_M which also occurs in the mass formulae for π and K. The term

$$\Delta_N = \frac{6}{F^2}(2K_1 + K_2 + \frac{4}{F^2}L_5\tau) \ ,$$

which describes the Zweig rule violating contributions of order $1/N_c$, only affects the quantities σ_1 and M_1^2. The coefficient M_1^2 contains a further $1/N_c$ correction which goes along with the topological susceptibility, $\overline{\tau} = \tau(1 - 12K_2/F^2)$.

Since the structure of the Lagrangian is the same as at leading order, the mass formula which interrelates M_η and $M_{\eta'}$ is of the same form, except that m_0, σ_0 must be replaced by m_1, σ_1,

$$M_\eta^2 = m_1^2 - \frac{\quad}{M_{\eta'}^2 \quad m_1^2} \tag{9}$$

The relation states that the current algebra formula $M_\eta^2 = \frac{1}{3}(4M_K^2 - M_\pi^2)$ receives two categories of SU(3) breaking corrections: While the first is governed by the same parameter Δ_M which also determines the corrections in the masses of the charged particles and is accounted for in the term m_1^2, the second arises from the η-η' transition matrix element of the operator $\bar{q}mq$ and is proportional to σ_1^2.

VII. Bounds

The relation (7) constrains the two ratios $x = m_u/m_d$ and $y = m_s/m_d$ to an ellipse. Since Q^2 is very large, the ellipse is well approximated by $x^2 + y^2/Q^2 = 1$, i.e. the center is at the origin and Q represents the large semiaxis, while the small one is equal to 1. To determine the individual ratios x, y one needs to know the quantity Δ_M, which describes the strength of SU(3) breaking in the mass formulae. According to eq.(6), the correction involves the two effective coupling constants L_5 and L_8. The former is known from the observed asymmetry in the decay constants F_π, F_K, but, as pointed out by Kaplan and Manohar [10], L_8 cannot be determined on purely phenomenologigal grounds. The scattering of the chiral perturbation theory results for the mass ratios m_u/m_d and m_s/m_d encountered in the literature [12]–[20] originates in this problem. Treating L_8 as a free parameter, one may obtain any value for Δ_M and thus reach any point on the ellipse. In particular, the possibility that the mass of the lightest quark might vanish is widely discussed in the literature, because this would remove the strong CP problem [21]. This possibility corresponds to $x = 0$, $y = Q$ and requires that the correction is large and negative, $\Delta_M = M_K^2/M_\pi^2/(Q + \frac{1}{2}) - 1 \simeq -0.45$.

I now wish to show that the framework specified above leads to a stringent lower bound on Δ_M which requires m_u to be different from zero. Admittedly, the hypothesis that the first few terms of the large N_c expansion yield a decent approximation for the theory of physical interest, $N_c = 3$, goes beyond solid phenomenology. This hypothesis, however, represents the only coherent theoretical explanation of the fact that the Okubo-Zweig-Iizuka-rule holds to a good approximation and I see no reason to doubt its reliability in the present case.

The bound arises from the fact that the relation (9) only admits a solution if $M_\eta^2 < m_1^2$, or

$$\Delta_M > -\frac{4M_K^2 - 3M_\eta^2 - M_\pi^2}{4(M_K^2 - M_\pi^2)} = -0.07 \tag{10}$$

The large negative value of Δ_M required if m_u were to vanish violates this condition. The inequality immediately follows from the trivial statement that η-η' mixing leads to a repulsion of the two levels. Although the derivation relies on the large N_c expansion and only holds up to terms of next-to-next-to-leading order in $1/N_c$, it is difficult to imagine how the higher order terms could upset the simple picture which underlies the above inequality.

I conclude that m_u is different from zero. As is well-known, the quantity $e^{i\theta}\det m$ then represents a physically significant parameter of the QCD Lagrangian. Why the mechanism which generates m and θ happens to be such that, to a high degree of accuracy, this quantity is real, represents a well-known puzzle [21], which cannot be solved the easy way, with $m_u = 0$.

A similar inequality also holds for the $1/N_c$ correction,

$$\Delta_N > 3\frac{(\sqrt{3}+1)M_\eta^2 - (\sqrt{3}-1)M_{\eta'}^2 - 2M_\pi^2}{8(M_K^2 - M_\pi^2)} = 0.18 \tag{11}$$

It indicates that Zweig rule violating effects cannot entirely be neglected.

As it stands, the mass formula (9) only determines the magnitude of the SU(3) asymmetry $\Delta_M \sim (m_s - \hat{m})$ if the size of the Zweig rule violation $\Delta_N \sim 1/N_c$ is known and vice versa. One may, e.g., choose a value for Δ_N, such that Δ_M saturates the inequality (10). This is the case, however, only if σ_1 vanishes, i.e. if $1 + \Delta_N - \Delta_M = 0$. In other words, the corrections would have to cancel the leading term. It is clear that in such a situation, the above formulae are meaningless. A coherent picture only results if both $|\Delta_M|$ and $|\Delta_N|$ are small compared to unity. The mass formula (9) shows that a negative value of the SU(3) asymmetry Δ_M requires exorbitant $1/N_c$ corrections. Even $\Delta_M = 0$ calls for large Zweig rule violations, $\Delta_N \simeq \frac{1}{2}$. The term $(1+\Delta_M-\Delta_N)^2$, which corrects the contribution generated by the singlet-octet transitions for effects of first nonleading order, would then instead modify this contribution by a factor of 4 if evaluated as it stands, eliminating it altogether if expanded to first order in the "corrections". The condition

$$\Delta_M > 0 \tag{12}$$

thus represents a generous lower bound for the region where a truncated $1/N_c$ expansion leads to meaningful results. It states that the current algebra formula, which relates the quark mass ratio $m_s/\hat{m}$ to the meson mass ratio M_K^2/M_π^2 represents an upper limit, $m_s/\hat{m} < 2M_K^2/M_\pi^2 - 1$. The corresponding bounds on the two ratios m_u/m_d and m_s/m_d depend on the value of Q,

$$\frac{m_u}{m_d} > \frac{M_\pi^4 Q^2 - M_K^4 + M_K^2 M_\pi^2}{M_\pi^4 Q^2 + M_K^4 - M_K^2 M_\pi^2} \,, \tag{13}$$

$$\frac{m_s}{m_d} < \frac{M_\pi^2 (2M_K^2 - M_\pi^2) Q^2}{M_\pi^4 Q^2 + M_K^4 - M_K^2 M_\pi^2} \,. \tag{14}$$

If Q is taken from the Dashen theorem, these relations state that the Weinberg ratios [13] correspond to the limiting case where the bounds are saturated, $m_u/m_d > 0.55, m_s/m_d < 20.1$. If the value of Q is lowered to $Q \simeq 22$, the bound on m_u/m_d becomes weaker, $m_u/m_d > 0.48$, while the one for m_s/m_d is sharpened to $m_s/m_d < 19.2$.

The estimate $m_u/m_d = 0.3 \pm 0.1$ obtained in ref.[17, 20] from a multipole analysis of the transitions $\psi' \to \psi\pi^0$, $\psi' \to \psi\eta$ is outside this range and is thus not consistent with the above arguments.

To demonstrate that the observed mass pattern is perfectly consistent with the hypothesis that both categories of symmetry breaking effects only generate small corrections, I recall that the contribution from the second term in eq. (9) is small, because it is suppressed by a factor of order $M_\eta^2/M_{\eta'}^2$. The corrections to that term are reduced by the same factor. The only new term in the second order formula, which does not get suppressed is the one which makes the difference between m_1^2 and m_0^2. Retaining only this term, the second order formula simplifies to

$$\Delta_M = -\frac{4M_K^2 - M_\pi^2 - 3M_\eta^2}{4(M_K^2 - M_\pi^2)} + \frac{2(M_K^2 - M_\pi^2)}{3M_{\eta'}^2 + M_\pi^2 - 4M_K^2} \,. \tag{15}$$

Numerically, this gives $\Delta_M = 0.18$, thus requiring a breaking of SU(3) symmetry in the mass formulae of the same order of magnitude as in the decay constants. The corresponding value for the $1/N_c$ correction is also quite small, $\Delta_N = 0.24$. I repeat, however, that the above framework does not predict the values of the two corrections individually, but only correlates them. In particular, the truncated large N_c expansion is also consistent with a somewhat smaller breaking of SU(3) symmetry and a correspondingly larger violation of the Zweig rule.

VIII. The decay $\eta \to 3\pi$

The breaking of chiral symmetry generated by the quark masses also shows up in the decay $\eta \to 3\pi$. In fact, this process represents a very sensitive probe for the quark mass matrix. Bose statistics does not allow three pions to form a configuration where the total angular momentum and the total isospin both vanish. The transition thus violates isospin symmetry. The transition amplitude contains terms proportional to the quark mass difference $m_d - m_u$ as well as contributions $\propto e^2$, generated by the electromagnetic interaction. The latter are strongly suppressed by chiral symmetry — the transition is due almost exclusively to the isospin breaking part $\frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)$ of the QCD Hamiltonian.[3] Current algebra implies that the matrix element relevant for the transition $\eta \to \pi^+\pi^-\pi^0$ is given by

$$A = -\epsilon_0 \frac{1}{F^2}\left(s - \tfrac{4}{3} M_\pi^2\right) , \qquad \epsilon_0 \equiv \frac{\sqrt{3}}{4} \frac{(m_d - m_u)}{(m_s - \hat{m})} .$$

The amplitude is linear in s and contains an Adler zero outside the physical region, at $s = \frac{4}{3}M_\pi^2$. This feature is clearly confirmed by the observed Dalitz plot distribution.

The decay amplitude is proportional to $(m_d - m_u)/(m_s - \hat{m})$, with a known factor of proportionality. Hence the observed value of the rate may be used to measure this ratio, which represents the strength of isospin breaking compared to the strength of SU(3)-breaking. The same quantity also determines the ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$ of meson masses. So,

[3]For a review and references to the literature, see [22].

chiral symmetry predicts the rate of η-decay in terms of the pseudoscalar masses. There is more to it, for two reasons.

(a) The mass formula (5), which relates the above ratio of meson masses to a ratio of quark masses, refers to pure QCD and neglects the electromagnetic self energies. The Dashen theorem states that, in the chiral limit, the self energies of K^0 and π^0 vanish, while the contributions to $M_{K^+}^2$ and $M_{\pi^+}^2$ are the same [11]. Moreover, the difference $(M_{\pi^+}^2 - M_{\pi^0}^2)_{\rm QCD}$ nearly vanishes, because it is of order $(m_d - m_u)^2$. Hence $(M_{K^0}^2 - M_{K^+}^2)_{\rm QCD} \simeq M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2$. Inserting the observed masses, this leads to the value $Q \simeq 24$ mentioned in section VI. The Dashen relation – which is a theorem only in the chiral limit – may, however, receive significant corrections [23], because some of the higher order contributions are enhanced by small energy denominators from low lying resonances. These tend to increase the electromagnetic self energy of the charged kaon and thus to reduce the value of Q. The problem is currently under more detailed study [24]. In the case of η-decay, the electromagnetic contributions are suppressed, so that the uncertainties connected with these do not play a significant role.

(b) The current algebra formula given above is valid as it stands only in the chiral limit, $m_u, m_d, m_s \to 0$. To analyze the implications of the observed decay rate for the relative magnitude of the quark masses, one needs to understand the size of the corrections to that relation. These are not the same as those appearing in the mass formula for $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$. In particular, the interference with the η' plays an important role in the matrix element of the transition $\eta \to 3\pi$, but does not show up in that mass ratio. The comparison of the two quantities thus also tests our understanding of the phenomena related to the U(1) anomaly and to the η'.

The first order corrections to the current algebra prediction for η-decay (chiral perturbation theory to one loop) are known [22]. Normalizing the amplitude with the kaon mass difference (e.m. self energies removed) and with the pion matrix element of the axial current,[4] the

[4]Since the expression accounts for the corrections of order m, one needs to distinguish between the constant F in the effective Lagrangian and the observed decay constants F_π, F_K, which differ from F through contributions of order m.

result is of the form

$$A = -\frac{(M^2_{K^0} - M^2_{K^+})_{\rm QCD}}{3\sqrt{3}\,F_\pi^2}\, M(s,t,u) \ , \tag{16}$$

where $M(s,t,u)$ is an explicitly known function, which exclusively contains measured quantities.

The one loop result does account for the mixing between η and η', but only to leading order. The interference of the two states also generates contributions at higher orders of the chiral expansion. Their size is determined by the ratio $\langle\eta'|\bar{q}mq|\eta\rangle/(M^2_{\eta'} - M^2_\eta)$, which compares the transition matrix element with the relevant energy denominator. Since the latter is relatively small, the higher order contributions might give rise to rather large corrections [25].

The one loop calculation is based on $\mathrm{SU}(3)_R\times\mathrm{SU}(3)_L$ and thus only involves the degrees of freedom of the pseudoscalar octet. In this framework, the η' only manifests itself indirectly, through its contributions to the effective coupling constants, like all other states which remain massive in the chiral limit, e.g. the ρ. In the calculation of the transition amplitude, the η' hides in the effective coupling constant L_7, which also occurs if the mass of the η is calculated within the same framework. The explicit expression for $M(s,t,u)$ contains the coupling constant L_7 through a correction term which is proportional to the deviation from the Gell-Mann-Okubo formula and is denoted by

$$\Delta_{\rm GMO} \equiv \frac{4M_K^2 - 3M_\eta^2 - M_\pi^2}{M_\eta^2 - M_\pi^2} \ .$$

At one loop order of chiral perturbation theory, the mixing with the η' thus affects the decay amplitude through precisely the same term which also shows up in the mass spectrum of the pseudoscalars. Dropping all other terms and disregarding contributions of order M_π^2, the one loop result reduces to

$$M(s,t,u) = \frac{3s}{M_\eta^2 - M_\pi^2}(1 + \tfrac{2}{3}\Delta_{\rm GMO}) + \ldots = \frac{4s}{M_K^2 - M_\pi^2}(1 + \Delta_{\rm GMO}) + \ldots$$

where I have used the identity $(M_\eta^2 - M_\pi^2)(1+\frac{1}{3}\Delta_{\rm GMO}) = \frac{4}{3}(M_K^2 - M_\pi^2)$. This shows that the main effect of the coupling constant which is connected with $\eta\eta'$ mixing is to multiply the current algebra amplitude with the factor $(1+\Delta_{\rm GMO}) \simeq 1.22$, a correction of reasonable size, characteristic for SU(3) breaking effects.

IX. $\eta'\eta$ mixing in the decay amplitude

The chiral perturbation theory result appears to be at variance with the direct calculation of the mixing effect, which runs as follows [25]. The effective Lagrangian discussed in the preceding sections, which involves the degrees of freedom of the η as well as those of the η', is perfectly suited for the purpose. Working at leading order, i.e. retaining only the first line in eq.(1), the tree graph relevant for the decay is readily worked out. It is of the same form as the current algebra prediction,

$$A = -\epsilon\,\frac{1}{F_\pi^2}(s - s_A)$$

and merely differs in the values of the two constants ϵ, s_A. Since the Adler condition implies that s_A is negligibly small, it suffices to calculate the coefficient of the term proportional to s. As in the current algebra calculation, the relevant vertex arises from the standard four-pion interaction $\propto \mathrm{tr}\{[\partial_\mu \pi, \pi][\partial^\mu \pi, \pi]\}$. The only difference produced by the occurrence of a singlet field is that the mixing angles — which relate the neutral fields to the eigenstates of the mass matrix — are modified: There are now three neutral fields $\varphi^3, \varphi^8, \varphi^9$ and three neutral particles π^0, η, η'. The coefficient of the term linear in s is the angle ϵ occurring in the relation $\pi^3 = \pi^0 - \epsilon\,\eta - \epsilon'\,\eta'$. The standard current algebra Lagrangian yields $\epsilon = \epsilon_0$. The interference with the η' generates a correction, given by [25]

$$\epsilon = \epsilon_0\,\frac{\cos\theta_{\eta'\eta} - \sqrt{2}\,\sin\theta_{\eta'\eta}}{\cos\theta_{\eta'\eta} + \frac{1}{\sqrt{2}}\,\sin\theta_{\eta'\eta}}\,\cos\theta_{\eta'\eta}\ . \tag{17}$$

Inserting the value of the mixing angle which follows from eq.(2) this formula gives $\epsilon \simeq 2\epsilon_0$. So, mixing with the η' appears to increase the amplitude by a factor of 2 rather than by the modest correction found above. If this were correct, one would have to conclude that the singularity due to the η' strongly distorts the decay amplitude and that chiral perturbation theory could not be trusted here.

To identify the origin of the problem, I express the above angular factor in terms of the masses of the particles. Eliminating the mass of the η' with $(m_0^2 - M_\eta^2)(M_{\eta'}^2 - m_0^2) = \sigma_0^2$, the relation (2) for the mixing angle takes the form $\mathrm{tg}\theta_{\eta'\eta} = (M_\eta^2 - m_0^2)/\sigma_0$. The angular factor thus becomes

$$\frac{\cos\theta_{\eta'\eta} - \sqrt{2}\sin\theta_{\eta'\eta}}{\cos\theta_{\eta'\eta} + \frac{1}{\sqrt{2}}\sin\theta_{\eta'\eta}} = \frac{2(2M_K^2 - M_\eta^2 - M_\pi^2)}{M_\eta^2 - M_\pi^2} \equiv 1 + \Delta_{\mathrm{G}}$$

This is remarkable, because it shows that the result of the above direct calculation of the effect of mixing on the decay amplitude may equally well be written as

$$\epsilon = \epsilon_0 \left\{1 + \Delta_{\mathrm{GMO}}\right\} \cos\theta_{\eta'\eta} \ . \tag{18}$$

As mentioned above, the experimental value of $1 + \Delta_{\mathrm{GMO}}$ is 1.22 and the cosine factor even reduces the number a litte, while in the form written earlier, the same result gave $\epsilon/\epsilon_0 \simeq 2$.

The discrepancy illustrates the discussion of section V: If the terms of order m^2 are dropped, the effective Lagrangian in eq. (1) implies that the pseudoscalar masses should show strong departures from the Gell-Mann-Okubo formula. The observed mass pattern is perfectly consistent with the assumption that the corrections to the leading term in the effective Lagrangian are small, but they definitely are different from zero. The main point here is that these corrections necessarily also affect the amplitude of the transition $\eta \to 3\pi$. In fact, the result of the chiral perturbation theory calculation amounts to a low energy theorem: To order p^4 the slope of the decay amplitude involves precisely the same combination of these couplings which determine the deviation from the Gell-Mann-Okubo formula. This is why, taken by itself, the direct calculation sketched above does not yield a decent estimate for ϵ, unless the result is written in the form (18), where it differs from the chiral perturbation theory result only by a factor of $\cos\theta_{\eta'\eta}$. In the chiral expansion, this factor represents a correction of order m_s^2 and is beyond the accuracy of the calculation. It is incorrect to multiply the one loop formula with the enhancement factor occurring in eq.(17).

Stated otherwise, the effective coupling constant L_7, which accounts for the mixing of the pseudoscalar octet with the η', is not the only term which matters here – L_5 and L_8 also contribute, both to the masses of the pseudoscalar octet and to the transition amplitude. In the framework of pole models [9], these couplings are dominated by the exchange of scalar particles with a mass comparable to $M_{\eta'}$. In contrast to the above calculation, which only accounts for the effects due to the η', the one loop result of chiral perturbation theory accounts for all of the contributions to order p^4. It shows that the various couplings contribute with different signs and that the net effect is quite small.

Note that these statements need not hold for radiative transitions, which may very well be distorted by the pole due to η'-exchange. The 3π channel is special, because the transition amplitude is determined by the effective Lagrangian of the strong interaction — this is why it is firmly tied to the mass spectrum of the pseudoscalars.

X. Dispersive analysis of η-decay

The motivation for discussing the chiral perturbation theory predictions for the transition $\eta \to 3\pi$ in such detail is that this process is a very valuable source of information for a particular ratio of quark masses. I conclude this paper with a brief sketch of ongoing work aimed at a determination of this ratio from the observed decay rate [26, 27].

The one loop formula (16) for the decay amplitude may equivalently be written as

$$A = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}\, M(s,t,u) \ ,$$

with $Q^2 = (m_s^2 - \hat{m})^2/(m_d^2 - m_u^2)$. A measurement of the decay rate thus amounts to a measurement of Q. Apart from experimental uncertainties, the accuracy of the result for Q^2 is determined by the accuracy of the theoretical prediction for the dimensionless factor $M(s,t,u)$.

A rough estimate of the uncertainties to be attached to the one loop result is obtained by comparing the size of the corrections with the current algebra formula. The corrections depend on the energy of the decay products. The amplitude still exhibits a zero in the vicinity of the two points $s = u = \frac{4}{3}M_\pi^2$ and $s = t = \frac{4}{3}M_\pi^2$. There, the corrections are small.

The branch point singularity at $s = 4M_\pi^2$, which is generated by the final state interaction between the two charged pions, however, produces a significant enhancement of the correction, which strongly grows with s. At the center of the Dalitz plot, where $\sqrt{s} \simeq 345\,\mathrm{MeV}$, the correction amounts to 50 % of the leading term, indicating that (i) the current algebra formula underestimates the decay rate by a significant factor and (ii) the first two terms of the chiral perturbation series do not yield an adequate representation of the amplitude in the entire physical region.

The reason why in part of the physical region, the corrections are unusually large is understood. Chiral symmetry implies that the pions are subject to an interaction with a strength determined by F_π. In the chiral limit, the corresponding I = 0 S-wave phase shift is given by $\delta_0^0 = s/16\pi F_\pi^2$ and thus grows with the square of the center of mass energy. In the present case, this means that the corrections rapidly grow with the energy of the charged pion pair.

There is a well-known method which allows one to calculate the final state interaction effects, also if they are large: dispersion relations, Indeed, this method has been applied to the decay $\eta \to 3\pi$ long ago [28] and there are several more recent papers on the topic [29] Unitarity determines the discontinuities across the various cuts in terms of the corresponding scattering amplitude. Allowing for sufficiently many subtractions, only the low energy region makes a significant contribution to the dispersion integrals, such that only the two-particle cuts generated by elastic scattering need be considered. For the same reason, the partial wave expansion may be restricted to the S- and P-waves.

In fact, the phase shifts may be determined with the same technique. A thorough analysis of that problem was carried out some time ago, on the basis of the Roy equations [30]. In that case, the available data lead to a remarkably accurate determination of the S- and P-waves in terms of only two subtraction constants, which may be identified with the two S-wave scattering lengths. Since chiral perturbation theory predicts the scattering lengths to within small uncertainties, the $\pi\pi$-scattering amplitude is known rather accurately throughout the low energy region [31]. Moreover, there are rigorous sum rules and inequalities which strongly interrelate the threshold parameters [32]. The comparison with the recently completed calculation of the $\pi\pi$ scattering amplitude in chiral perturbation theory to two loops [33] and with the dispersive analysis in [34] will shed a considerable amount of light on the issue, in particular also on the accuracy within which the subtraction constants can be determined in this particular case.

The main point here is that chiral perturbation theory is needed only for the subtraction constants. The dependence of the amplitude on the kinematic variables then follows from analyticity and unitarity. The imaginary part of the one loop amplitude is determined by current algebra. Analyticity then fixes the real part, up to subtractions. In the case of η decay, the dispersive representation of the one loop amplitude contains three subtraction constants. The representation is of the form $M(s,t,u) = a + b\,s + c\,(s^2 + 2\,t\,u) + D(s,t,u)$, where $D(s,t,u)$ is a sum of subtracted dispersion integrals over the imaginary part and is fully determined by the latter. The constants a and b already occur in the current algebra formula. The one loop representation determines their values up to and including corrections of order m. The new term c is related to the coupling constant L_3.

The recent dispersive analysis of η-decay described in [26, 27] confirms the rough estimate of the higher order final state interaction effects given in [22]. If the three subtraction constants are known, the uncertainties in the result for the rate are less than 10%. Since the rate is inversely proportional to Q^4, this corresponds to an uncertainty in Q of order $\Delta Q = \pm 0.5$.

The main limitation of the dispersive method does not arise from the uncertainties in the dispersion integrals, but from those in the subtraction constants. The constant a determines the position of the Adler zero, which is protected by $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ and does therefore not give rise to a significant uncertainty. The term c does not represent a problem, either. In the low energy region, this term only represents a small correction, determined by the coupling constant L_3, whose value is known quite accurately from $K_{\ell 4}$ decay [35]. Moreover, the observed Dalitz plot distribution may be used to directly determine the value of c and to check the prediction which relates this term to the analogous contribution in $K_{\ell 4}$ decay.

The crucial term is the constant b, which determines the overall normalization of the amplitude. This is the term discussed in the preceding section, where I argued that the interference with the η' does not significantly affect the one loop result for this constant. Since the contribution from b enters the amplitude through the factor b/Q^2, the corresponding uncertaintes in the result for Q are smaller by a factor of two. As discussed above, the current algebra prediction $b = 3/(M_\eta^2 - M_\pi^2)$ receives known corrections of order m, determined by Δ_{GMO}. Expressed in terms of Q, these correspond to the shift $Q \rightarrow (1 + \frac{1}{3}\Delta_{GMO})Q \simeq 1.07\,Q$. The higher order effects should be significantly smaller. The cosine factor in eq.(18), which accounts for the effects of order m^2 from the interference with the η', reduces the result for Q by $\sqrt{\cos\theta_{\eta'\eta}} \simeq 0.96$. Taking this as an estimate for the size of the higher order terms and including the errors from the dispersion integrals, I conclude that η-decay allows a measurement of Q within an uncertainty of $\Delta Q = \pm 1$.

Unfortunately, the experimental situation is not clear [36]. The value of $\Gamma_{\eta\rightarrow\pi^+\pi^-\pi^0}$ relies on the rate of the decay into two photons. The two different methods of measuring $\Gamma_{\eta\rightarrow\gamma\gamma}$ — photon-photon-collisions and Primakoff effect — yield conflicting results. Expressed in terms of Q, the discrepancy amounts to $\Delta Q = 2.5$, one of the few cases in strong interaction physics where the theoretical results are more accurate than the experimental ones.

While the data based on the Primakoff effect lead to a value which is in perfect agreement with the number $Q \simeq 24$ quoted above, the $\gamma\gamma$ data yield a significantly lower result. The former implies that the corrections to the Dashen theorem are small, while the latter amounts to the opposite. The average value given by the particle data group is dominated by the $\gamma\gamma$ data. The experimental information thus indicates a value like $Q \simeq 22$ and thereby confirms the conclusions reached in [23]. Hopefully, the experimental situation will be clarified.

XI. Conclusion

In conclusion, I list the consequences of the above considerations for the ratios of the light quark masses.

Q	24	24	22	22
Δ_M	0	0.18	0	0.18
m_u/m_d	0.55	0.66	0.49	0.61
m_s/m_d	20.1	18.8	19.2	17.5
$m_s/\hat{m}$	26	22	26	22
R	43	51	36	42

The first column contains the current algebra values and the second shows the numbers which result if Δ_M is taken from the large N_c estimate in eq.(6) while Q is left unchanged. Since the correction Δ_M is positive, the values of m_s/m_d and $m_s/\hat{m}$ are lowered, while m_u/m_d and $R \equiv (m_s - \hat{m})/(m_d - m_u)$ increase. The third entry shows the effect of the change in Q discussed in the preceding section, taken by itself. The number for $m_s/\hat{m}$ stays unchanged , and the same is true, approximately, for m_s/m_d, while m_u/m_d and R decrease. Finally, the last entry lists the results if both of these changes are accounted for. This column represents the net result of the present paper in numerical form.

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